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# The Complete Hyperbolicity of Cylindric Billiards

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**Abstract.** The connected configuration space of a so called cylindric billiard system is a flat torus minus finitely many spherical cylinders. The dynamical system describes the uniform motion of a point particle in this configuration space with specular reflections at the boundaries of the removed cylinders. It is proven here that under a certain geometric condition — slightly stronger than the necessary condition presented in [S-Sz(1998)] — a cylindric billiard flow is completely hyperbolic. As a consequence, every hard ball system is completely hyperbolic — a result strengthening the theorem of [S-Sz(1999)].

## 1. Introduction

Non-uniformly hyperbolic systems (possibly, with singularities) play a pivotal role in the ergodic theory of dynamical systems. Their systematic study started several decades ago, and it is not our goal here to provide the reader with a comprehensive review of the history of these investigations but, instead, we opt for presenting in nutshell a cross section of a few selected results.

In 1939 G. A. Hedlund and E. Hopf [He(1939)], [Ho(1939)], proved the hyperbolic ergodicity of geodesic flows on closed, compact surfaces with constant negative curvature by inventing the famous method of "Hopf chains" constituted by local stable and unstable invariant manifolds.

In 1963 Ya. G. Sinai [Sin(1963)] formulated a modern version of Boltzmann's ergodic hypothesis, what we call now the "Boltzmann-Sinai ergodic hypothesis": the billiard system of $N$ $(\geq 2)$ hard balls of unit mass moving in the flat torus $\mathbb{T}^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu$ $(\nu \geq 2)$ is ergodic after we make the standard reductions by fixing the values of the trivial invariant quantities. It took seven years until he proved this conjecture for the case $N = 2$, $\nu = 2$ in [Sin(1970)]. Another 17 years later N. I. Chernov and Ya. G. Sinai [S-Ch(1987)] proved the hypothesis for the case $N = 2$, $\nu \geq 2$ by also proving a powerful and very useful theorem on local ergodicity.

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In the meantime, in 1977, Ya. Pesin [P(1977)] laid down the foundations of his theory on the ergodic properties of smooth, hyperbolic dynamical systems. Later on this theory (nowadays called Pesin theory) was significantly extended by A. Katok and J-M. Strelcyn [K-S(1986)] to hyperbolic systems with singularities. That theory is already applicable for billiard systems, too.

Until the end of the seventies the phenomenon of hyperbolicity (exponential unstability of the trajectories) was almost exclusively attributed to some direct geometric scattering effect, like negative curvature of space, or strict convexity of the scatterers. This explains the profound shock that was caused by the discovery of L. A. Bunimovich [B(1979)]: certain focusing billiard tables (like the celebrated stadium) can also produce complete hyperbolicity and, in that way, ergodicity. It was partly this result that led to Wojtkowski's theory of invariant cone fields, [W(1985)], [W(1986)].

The big difference between the system of two balls in $\mathbb{T}^\nu$ ($\nu \geq 2$, [S-Ch(1987)]) and the system of $N$ ($\geq 3$) balls in $\mathbb{T}^\nu$ is that the latter one is merely a so called semi-dispersive billiard system (the scatterers are convex but not strictly convex sets, namely cylinders), while the former one is strictly dispersive (the scatterers are strictly convex sets). This fact makes the proof of ergodicity (mixing properties) much more complicated. In our series of papers jointly written with A. Krámli and D. Szász [K-S-Sz(1990)], [K-S-Sz(1991)], and [K-S-Sz(1992)] we managed to prove the (hyperbolic) ergodicity of three and four billiard balls in the toroidal container $\mathbb{T}^\nu$. By inventing new topological methods and the Connecting Path Formula (CPF), in my two-part paper [Sim(1992)] I proved the (hyperbolic) ergodicity of $N$ hard balls in $\mathbb{T}^\nu$, provided that $N \leq \nu$.

The common feature of hard ball systems is — as D. Szász pointed this out first in [Sz(1993)] and [Sz(1994)] — that all of theom belong to the family of so called cylindric billiards, the definition of which can be found later in this paragraph. However, the first appearance of a special, 3-D cylindric billiard system took place in [K-S-Sz(1989)], where we proved the ergodicity of a 3-D billiard flow with two orthogonal cylindric scatterers. Later D. Szász [Sz(1994)] presented a complete picture (as far as ergodicity is concerned) of cylindric billiards with cylinders whose generator subspaces are spanned by mutually orthogonal coordinate axes. The task of proving ergodicity for the first non-trivial, non-orthogonal cylindric billiard system was taken up in [S-Sz(1994)].

Finally, in our joint venture with D. Szász [S-Sz(1999)] we managed to prove the complete hyperbolicity of *typical* hard ball systems.

**1.1. Cylindric billiards.** Consider the $d$-dimensional ($d \geq 2$) flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ supplied with the usual Riemannian inner product $\langle\, .\, ,\, .\, \rangle$ inherited from the standard inner product of the universal covering space $\mathbb{R}^d$. Here $\mathcal{L} \subset \mathbb{R}^d$ is supposed to be a lattice, i. e. a discrete subgroup of the additive group $\mathbb{R}^d$ with $\text{rank}(\mathcal{L}) = d$. The reason why we want to allow general lattices other than just the integer lattice $\mathbb{Z}^d$ is that otherwise the hard ball systems would not be covered! The geometry of the structure lattice $\mathcal{L}$ in the case of a hard ball system is significantly different from the geometry of the standard lattice $\mathbb{Z}^d$ in the standard Euclidean space $\mathbb{R}^d$, see subsection 2.4, especially (2.4.2) and (2.4.5).

The configuration space of a cylindric billiard is $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \dots \cup C_k)$, where the cylindric scatterers $C_i$ ($i = 1, \dots, k$) are defined as follows:

Let $A_i \subset \mathbb{R}^d$ be a so called lattice subspace of $\mathbb{R}^d$, which means that $\text{rank}(A_i \cap$

$\mathcal{L}) = \dim A_i$. In this case the factor $A_i/(A_i \cap \mathcal{L})$ is a subtorus in $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ which will be taken as the generator of the cylinder $C_i \subset \mathbb{T}^d$, $i = 1, \dots, k$. Denote by $L_i = A_i^{\perp}$ the orthocomplement of $A_i$ in $\mathbb{R}^d$. Throughout this article we will always assume that $\dim L_i \geq 2$. Let, furthermore, the numbers $r_i > 0$ (the radii of the spherical cylinders $C_i$) and some translation vectors $t_i \in \mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ be given. The translation vectors $t_i$ play a crucial role in positioning the cylinders $C_i$ in the ambient torus $\mathbb{T}^d$. Set

$$C_i = \left\{ x \in \mathbb{T}^d :\ \operatorname{dist}\left(x - t_i, A_i/(A_i \cap \mathcal{L})\right) < r_i \right\}.$$

In order to avoid further unnecessary complications, we always assume that the interior of the configuration space $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \dots \cup C_k)$ is connected. The phase space $\mathbf{M}$ of our cylindric billiard flow will be the unit tangent bundle of $\mathbf{Q}$ (modulo some natural glueings at its boundary), i. e. $\mathbf{M} = \mathbf{Q} \times \mathbb{S}^{d-1}$. (Here $\mathbb{S}^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$.)

The dynamical system $(\mathbf{M}, \{S^t\}_{t \in \mathbb{R}}, \mu)$, where $S^t$ $(t \in \mathbb{R})$ is the dynamics defined by uniform motion inside the domain $\mathbf{Q}$ and specular reflections at its boundary (at the scatterers), and $\mu$ is the Liouville measure, is called a cylindric billiard flow we want to investigate. (As to notions and notations in connection with semi-dispersive billiards, the reader is kindly recommended to consult the work [K-S-Sz(1990)].)

**Transitive cylindric billiards.**

The main conjecture concerning the (hyperbolic) ergodicity of cylindric billiards is the "Erdőtarcsa conjecture" (named after the picturesque village in rural Hungary where it was initially formulated) that appeared as Conjecture 1 in Section 3 of [S-Sz(1998)]:

**The Erdőtarcsa conjecture.** A cylindric billiard flow is ergodic if and only if it is transitive. (As for the definition and basic features of transitivity, see Section 3 (especially between 3.1 and 3.6) of [S-Sz(1998)] or subsection 2.2 below.) In that case the cylindric billiard system is actually a completely hyperbolic Bernoulli flow, see [C-H(1996)] and [O-W(1998)].

The theorem of this paper proves a slightly relaxed version of this conjecture (only full hyperbolicity without ergodicity) for a wide class of cylindric billiard systems, namely the so called "transverse systems" (see subsection 2.3 below) which include every hard ball system:

**Theorem.** Assume that the cylindric billiard system is transverse, see subsection 2.3. Then this billiard flow is completely hyperbolic, i. e. all relevant Lyapunov exponents are nonzero almost everywhere. Consequently, such dynamical systems have (at most countably many) ergodic components of positive measure, and the restriction of the flow to the ergodic components has the Bernoulli property, see [C-H(1996)] and [O-W(1998)].

**Corollary of the theorem.** Every hard ball system — necessarily being a transverse cylindric billiard system, see subsection 2.4 — is completely hyperbolic.

Thus, the theorem of this paper generalizes the main result of [S-Sz(1999)], where the complete hyperbolicity of *almost every* hard ball system was proven.

**Organizing of the paper.** After the technical preparation in Section 2, the theorem will be proven in the two subsequent sections. According to the usually accepted strategy developed in the series of papers [K-S-Sz(1989, 1991, 1992)], [Sim(1992)], the proof of full hyperbolicity should consist of two major steps:

**Step 1. (Geometric-algebraic considerations).** To prove that the existence of a combinatorially rich (appropriately defined!) trajectory segment $S^{[a,b]}x_0$ for a smooth phase point $x_0 \in \mathbf{M}$ implies (modulo some smooth, proper submanifolds) that the phase point $x_0$ is hyperbolic (or, using the older language, sufficient). This will be carried out in Section 3.

**Step 2. (Dynamical-topological part).** To show that for $\mu$-almost every phase point $x_0$ the symbolic collision sequence of the entire trajectory of $x_0$ is combinatorially rich. This will be accomplished in Section 4.

## 2. Prerequisites

**2.1. Sub-billiards.** Assume that a subset $I \subset \{1, \dots, k\}$ is given, and we consider the cylindric billiard flow in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ so that only the cylinders $\{C_i : i \in I\}$ are retained as scatterers; the other ones are no longer removed from the configuration space $\mathbf{Q}$ and the uniformly moving point $q = q(x)$ $(x = (q, v) \in \mathbf{M})$ can freely pass through them. We call the arising billiard flow a *sub-billiard*.

It turns out pretty soon that the name "factor billiard" would have been much better. Namely, let us consider the linear subspaces $E^+ = \text{span}\{L_i : i \in I\}$ and $E_0 = (E^+)^\perp = \bigcap_{i\in I} A_i$. It is an elementary exercise to show that the intersection $E_0$ of the lattice subspaces $A_i$ is also a lattice subspace, i. e. $\text{rank}(E_0 \cap \mathcal{L}) = \dim E_0$.

It is easy to see that the sub-billiard flow $\{S_I^t\}$ $(t \in \mathbb{R})$ defined by the scatterers $\{C_i : i \in I\}$ has the following peculiarity: the velocity component $P_{E_0}(v_t)$ of the moving phase point $x_t = (q_t, v_t)$ does not change, and in the direction of the subspace $E_0$ (or, equivalently, in the direction of the subtorus $E_0/(E_0 \cap \mathcal{L}) \subset \mathbb{R}^d/\mathcal{L}$) the motion of $q_t$ is conditionally periodic. (Here, as always, $P_{E_0}(\,.\,)$ denotes the orthogonal projection of $\mathbb{R}^d$ onto the subspace $E_0$.) According to the invariance of the quantity $P_{E_0}(v_t)$, we fix its value by introducing the reduction $P_{E_0}(v_t) = 0$. After this reduction the sub-billiard flow $\{S_I^t\}$ will have a translation invariance in the direction of the subtorus $E_0/(E_0 \cap \mathcal{L})$, thus we factorize out the configuration space with respect to spatial translations by elements $\tau \in E_0/(E_0 \cap \mathcal{L})$ as follows: $q \sim q' \iff q - q' \in E_0/(E_0 \cap \mathcal{L})$. The flow arising after the reduction $P_{E_0}(v) = 0$ and the above factorization is denoted by $\{S_I^t\}$. Let us describe now the natural configuration and velocity spaces of the flow $\{S_I^t\}$. The velocity space is obviously the orthocomplement $E^+$ of the lattice subspace $E_0 \subset \mathbb{R}^d$. (We note that the space $E^+$ does not have to be a lattice subspace!) After specifying the kinetic energy $\varepsilon = \frac{1}{2}||v||^2$ of the subsystem, we get the sphere of radius $\sqrt{2\varepsilon}$ in the Euclidean space $E^+$ as the velocity space for the sub-billiard flow $\{S_I^t\}$. As far as the configuration space $\mathbf{Q} = \mathbf{Q}_I$ is concerned, it is naturally the factor torus

$$\mathbb{T}^d/\left(E_0/(E_0 \cap \mathcal{L})\right) = \mathbb{R}^d/(\mathcal{L} + E_0)$$

(minus the intersections of the cylinders $\{C_i : i \in I\}$ with that factor torus) supplied with the Euclidean metric of the space $E^+$ as the Riemannian metric on $\mathbb{T}^d/\left(E_0/(E_0 \cap \mathcal{L})\right)$. Note that the subspace $E^+$ can be naturally identified with the tangent spaces of the factor torus $\mathbb{T}^d/\left(E_0/(E_0 \cap \mathcal{L})\right)$ at different points.

By projecting the whole space $\mathbb{R}^d$ onto $E^+$ we see that the factor torus

$$\mathbb{T}^d/\left(E_0/(E_0\cap\mathcal{L})\right)=\mathbb{R}^d/(\mathcal{L}+E_0)$$

can be naturally identified with the factor $E^+/P_{E^+}(\mathcal{L})$. We note that — as it follows easily from the fact that $E_0=(E^+)^\perp$ is a lattice subspace — the projection $P_{E^+}(\mathcal{L})$ of the lattice $\mathcal{L}$ onto $E^+$ is a lattice in the subspace $E^+$.

**2.2. Transitivity.** Let $L_1,\dots,L_k\subset\mathbb{R}^d$ be subspaces, $\dim L_i\ge 2$, $A_i=L_i^\perp$, $i=1,\dots,k$. Set

$$\mathcal{G}_i=\left\{U\in \mathrm{SO}(d):\, U|A_i=\mathrm{Id}_{A_i}\right\},$$

and let $\mathcal{G}=\langle\mathcal{G}_1,\dots,\mathcal{G}_k\rangle\subset \mathrm{SO}(d)$ be the algebraic generate of the compact, connected Lie subgroups $\mathcal{G}_i$ in $\mathrm{SO}(d)$. The following notions appeared in Section 3 of [S-Sz(1998)].

**Definition 2.2.1.** We say that the system of base spaces $\{L_1,\dots,L_k\}$ (or, equivalently, the cylindric billiard system defined by them) is *transitive* if and only if the group $\mathcal{G}$ acts transitively on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$.

**Definition 2.2.2.** We say that the system of subspaces $\{L_1,\dots,L_k\}$ has the Orthogonal Non-splitting Property (ONSP) if there is no non-trivial orthogonal splitting $\mathbb{R}^d=B_1\oplus B_2$ of $\mathbb{R}^d$ with the property that for every index $i$ $(1\le i\le k)$ $L_i\subset B_1$ or $L_i\subset B_2$.

The next result can be found in Section 3 of [S-Sz(1998)] (see 3.1–3.6 thereof):

**Proposition 2.2.3.** For the system of subspaces $\{L_1,\dots,L_k\}$ the following three properties are equivalent:

(1) $\{L_1,\dots,L_k\}$ is transitive;
(2) $\{L_1,\dots,L_k\}$ has the ONSP;
(3) the natural representation of $\mathcal{G}$ in $\mathbb{R}^d$ is irreducible.

**2.3. Transverseness.**

**Definition 2.3.1.** We say that the system of subspaces $\{L_1,\dots,L_k\}$ of $\mathbb{R}^d$ is *transverse* if the following property holds: For every *non-transitive* subsystem $\{L_i:\, i\in I\}$ $(I\subset\{1,\dots,k\})$ there exists an index $j_0\in\{1,\dots,k\}$ such that $P_{E^+}(A_{j_0})=E^+$, where $A_{j_0}=L_{j_0}^\perp$, and $E^+=\mathrm{span}\{L_i:\, i\in I\}$. We note that in this case, necessarily, $j_0\notin I$, otherwise $P_{E^+}(A_{j_0})$ would be orthogonal to the subspace $L_{j_0}\subset E^+$. Therefore, every transverse system is automatically transitive.

**2.4. A major family of examples.**

**2.4.1. Hard ball systems.** Hard ball systems in the standard unit torus $\mathbb{T}^\nu=\mathbb{R}^\nu/\mathbb{Z}^\nu$ $(\nu\ge 2)$ with positive masses $m_1,\dots,m_N$ are described (for example) in Section 1 of [S-Sz(1999)]. These are the dynamical systems describing the motion of $N$ $(\ge 2)$ hard balls with radius $r>0$ and positive masses $m_1,\dots,m_N$ in the standard unit torus $\mathbb{T}^\nu=\mathbb{R}^\nu/\mathbb{Z}^\nu$. The center of the $i$-th ball is denoted by $q_i$ $(\in\mathbb{T}^\nu)$, its time derivative is $v_i=\dot{q}_i$, $i=1,\dots,N$. One uses the standard reduction of kinetic energy $\varepsilon=\frac12\sum_{i=1}^N m_i||v_i||^2=\frac12$. The arising configuration space (still without the removal of the scattering cylinders $C_{i,j}$) is the torus

$$\mathbb{T}^{\nu N}=(\mathbb{T}^\nu)^N=\{(q_1,\dots,q_N):\, q_i\in\mathbb{T}^\nu,\ i=1,\dots,N\}$$

supplied with the Riemannian inner product

$$\langle v, v' \rangle = \sum_{i=1}^{N} m_i \langle v_i, v_i' \rangle \tag{2.4.2}$$

in its common tangent space $\mathbb{R}^{\nu N} = \left(\mathbb{R}^{\nu}\right)^N$. Now the Euclidean space $\mathbb{R}^{\nu N}$ with the inner product (2.4.2) plays the role of $\mathbb{R}^d$ in the original definition of cylindric billiards, see Section 1 above.

The generator subspace $A_{i,j} \subset \mathbb{R}^{\nu N}$ $(1 \le i < j \le N)$ of the cylinder $C_{i,j}$ (describing the collisions between the $i$-th and $j$-th balls) is given by the equation

$$A_{i,j} = \left\{ (q_1, \dots, q_N) \in \left(\mathbb{R}^{\nu}\right)^N :\ q_i = q_j \right\}, \tag{2.4.3}$$

see (4.3) in [S-Sz(1998)]. Its orthocomplement $L_{i,j} \subset \mathbb{R}^{\nu N}$ is then defined by the equation

$$L_{i,j} = \left\{ (q_1, \dots, q_N) \in \left(\mathbb{R}^{\nu}\right)^N :\ q_k = 0 \text{ for } k \ne i, j, \text{ and } m_i q_i + m_j q_j = 0 \right\}, \tag{2.4.4}$$

see (4.4) in [S-Sz(1998)]. Easy calculation shows that the cylinder $C_{i,j}$ is indeed spherical and the radius of its base sphere is equal to $r_{i,j} = 2r\sqrt{\frac{m_i m_j}{m_i + m_j}}$, see Section 4, especially formula (4.6) in [S-Sz(1998)].

The structure lattice $\mathcal{L} \subset \mathbb{R}^{\nu N}$ is clearly the integer lattice $\mathcal{L} = \mathbb{Z}^{\nu N}$.

Due to the presence of an extra invariant quantity $I = \sum_{i=1}^N m_i v_i$, one usually makes the reduction $\sum_{i=1}^N m_i v_i = 0$ and, correspondingly, factorizes the configuration space with respect to uniform spatial translations:

$$(q_1, \dots, q_N) \sim (q_1 + a, \dots, q_N + a), \quad a \in \mathbb{T}^{\nu},$$

see also subsection 2.1 above. The natural, common tangent space of this reduced configuration space is then

$$\mathcal{Z} = \left\{ (v_1, \dots, v_N) \in \left(\mathbb{R}^{\nu}\right)^N :\ \sum_{i=1}^{N} m_i v_i = 0 \right\} = \left( \bigcap_{i<j} A_{i,j} \right)^{\perp} = (\mathcal{A})^{\perp} \tag{2.4.5}$$

supplied again with the inner product (2.4.2), see also (4.1) and (4.2) in [S-Sz(1998)]. The base spaces $L_{i,j}$ of (2.4.4) are obviously subspaces of $\mathcal{Z}$, and we take $\tilde{A}_{i,j} = A_{i,j} \cap \mathcal{Z} = P_{\mathcal{Z}}(A_{i,j})$ as the orthocomplement of $L_{i,j}$ in $\mathcal{Z}$.

Note that the configuration space of the reduced system (with $\sum_{i=1}^N m_i v_i = 0$) is naturally the torus $\mathbb{R}^{\nu N}/(\mathcal{A} + \mathbb{Z}^{\nu N}) = \mathcal{Z}/P_{\mathcal{Z}}(\mathbb{Z}^{\nu N})$, see also subsection 2.1.

**Proposition 2.4.6.** For every hard ball system with parameters $N, \nu, r, m_1, \dots, m_N$ $(N, \nu \ge 2,\ m_i, r > 0)$ the collection of base spaces $\{L_{i,j} :\ 1 \le i < j \le N\}$ has the property of transverseness in the tangent space $\mathcal{Z}$.

**Proof.** Assume that $I \subset \{(i,j) :\ 1 \le i < j \le N\}$ is the index set of a nontransitive family $\{L_{i,j} :\ (i,j) \in I\}$ of base spaces in $\mathcal{Z}$. The set $I$ can be considered as the set of edges of a non-oriented collision graph $\mathcal{G}$ with vertex set $\{1, \dots, N\}$. It is shown in Remark 4.12 of [S-Sz(1998)] that the non-transitivity of $\{L_{i,j} :\ (i,j) \in I\}$ means that the graph $\mathcal{G}$ is not connected on the full vertex set $\{1, \dots, N\}$. Choose a pair $(i_0, j_0)$ $(1 \le i_0 < j_0 \le N)$ so that these indices belong to different connected components of $\mathcal{G}$. Then elementary consideration shows that $P_{E^+}(\tilde{A}_{i_0,j_0}) = E^+$, where $E^+ = \text{span}\{L_{i,j} :\ (i,j) \in I\}$. (As a matter of fact, specifying an element $q \in E^+$ means specifying the relative positions of the balls in each connected component of the graph $\mathcal{G}$. Finding a suitable element $\tilde{q} \in \tilde{A}_{i_0,j_0}$ with $P_{E^+}(\tilde{q}) = q$ precisely means that we ought to move each of the connected components of $\mathcal{G}$ uniformly in the ambient torus so that the centers of the $i_0$-th and $j_0$-th balls just coincide. However, this can obviously be accomplished.)

This finishes the proof of the proposition. $\square$

**2.5. Another family of examples: Connected "direct sum systems".**

Consider now such cylindric billiard systems in which the space $\mathbb{R}^d$ decomposes into a linear direct sum

$$\mathbb{R}^d = L_1 + L_2 + \cdots + L_k \tag{2.5.1}$$

of the base spaces $L_i$. With the decomposition (2.5.1) we associate a non-oriented graph $\mathcal{G}$ with the vertex set $\mathcal{V}(\mathcal{G}) = \{1, \dots, k\}$ and edge set

$$\mathcal{E}(\mathcal{G}) = \{\{i,j\} :\ i \ne j \text{ and } L_i \not\perp L_j\}.$$

It is then obvious that the transitivity of such a cylindric billiard system is equivalent to the connectedness of the graph of non-orthogonality $\mathcal{G}$ (on the full vertex set $\{1, \dots, k\}$), which we assume now.

**Proposition 2.5.2.** A "direct sum system" (described above) with a connected graph of non-orthogonality $\mathcal{G}$ enjoys the property of transverseness.

**Proof.** Assume that $I \subset \{1, \dots, k\}$, and the system of subspaces $\{L_i :\ i \in I\}$ is not transitive, i. e. $|I| < k$. Now, for any index $j_0 \in \{1, \dots, k\} \setminus I$ one has

$$L_{j_0} \cap \text{span}\,\{L_i :\ i \in I\} = L_{j_0} \cap E^+ = \{0\},$$

i. e. $\text{span}\left\{A_{j_0}, (E^+)^\perp\right\} = \mathbb{R}^d$ which, in turn, means that $P_{E^+}(A_{j_0}) = E^+$. $\square$

**Remark.** Consider a hard ball system with the graph of allowed collisions $\mathcal{G}$, see Remark 4.12 in [S-Sz(1998)]. Assume that the graph $\mathcal{G}$ is a tree, i. e. a connected graph without loop. It is an easy exercise to see that such a hard ball system belongs to the family of connected direct sum systems described above.

**Hyperbolic (sufficient) trajectories.** Their definition and fundamental properties can be found — for example — in Definition 2.12 and Lemma 2.13 of [K-S-Sz(1990)].

**2.7. The subsets $\mathbf{M}^0$ and $\mathbf{M}^\#$.** Denote by $\mathbf{M}^\#$ the set of all phase points $x \in \mathbf{M}$ for which the trajectory of $x$ encounters infintely many non-tangential collisions in both time directions. The trajectories of the points $x \in \mathbf{M} \setminus \mathbf{M}^\#$ are

lines: the motion is linear and uniform, see the appendix of [Sz(1994)]. It is proven in lemmas A.2.1 and A.2.2 of [Sz(1994)] that the closed set $\mathbf{M}\setminus\mathbf{M}^{\#}$ is a finite union of hyperplanes. Thus, in our study of complete hyperbolicity, we can discard the set $\mathbf{M}\setminus\mathbf{M}^{\#}$ and focus on the open set $\mathbf{M}^{\#}$ with full measure.

Denote by $\mathbf{M}^0$ the set of all non-singular phase points $x\in\mathbf{M}$, i. e. all phase points $x$ whose entire trajectory is smooth. Since the complement $\mathbf{M}\setminus\mathbf{M}^0$ of this set is a countable union of smooth, proper submanifolds of $\mathbf{M}$, we can again discard the zero set $\mathbf{M}\setminus\mathbf{M}^0$ and only consider phase points $x\in\mathbf{M}^0\cap\mathbf{M}^{\#}$.

**Finitely many collisions in finite time.** By the results of Vaserstein [V(1979)], Galperin [G(1981)] and Burago-Ferleger-Kononenko [B-F-K(1998)], in a semi-dispersive billiard flow there can only be finitely many collisions in finite time intervals, see Theorem 1.1 in [B-F-K(1998)]. Thus, the dynamics is well defined as long as the trajectory does not hit more than one boundary components at the same time.

## 3. Geometric-Algebraic Considerations.

We begin this section with some new notions. Consider the linear subspaces $A_i$ and $L_i=A_i^{\perp}$ ($i=1,\dots,k$, $\dim L_i\ge 2$) in $\mathbb{R}^d$ and the positive numbers (radii) $r_i$ associated with them. Furthermore, consider and fix a finite sequence $\Sigma=(\sigma(1),\dots,\sigma(m))$ of labels $\sigma(j)\in\{1,2,\dots,k\}$, a so called symbolic collision sequence.

**Definition 3.1.** We say that $\gamma$ is a *Euclidean path* with the collision sequence $\Sigma$ if the following properties hold:

(1) $\gamma:\,[0,\infty)\to\mathbb{R}^d$ is a piecewise linear, continuous curve in $\mathbb{R}^d$ with $\gamma(0)=0$;

(2) $\gamma$ has an arc length parametrization by $t$, i. e. $||\dot\gamma(t)||=1$ for $t\ge 0$;

(3) the velocity $\dot\gamma(t)$ has finitely many discontinuities and these discontinuities are jump discontinuities taking place at time moments $(0<)t_1<t_2<\dots<t_m<\infty$;

(4) the vectors of abrupt velocity change $\dot\gamma(t_j+0)-\dot\gamma(t_j-0)\ne 0$ belong to the base subspace $L_{\sigma(j)}$, $j=1,\dots,m$.

We can think of the curve $\gamma$ as the Euclidean lifting of a finite trajectory segment (extended to $t\to\infty$ with constant velocity, just for technical reasons) of the genuine cylindric billiard flow in $\mathbb{T}^d=\mathbb{R}^d/\mathcal{L}$. The $j$-th collision takes place at time moment $t_j$ at the boundary of the translated cylinder $a_j+C_{\sigma(j)}=a_j(\gamma)+C_{\sigma(j)}$ ($j=1,\dots,m$), where

$$
\begin{gathered}
C_i=\left\{x\in\mathbb{R}^d:\,\mathrm{dist}(x,A_i)<r_i\right\}\quad(i=1,\dots,k),\\
a_j=a_j(\gamma)=P_{\sigma(j)}\left(\gamma(t_j)\right)-r_{\sigma(j)}\cdot\frac{\dot\gamma(t_j+0)-\dot\gamma(t_j-0)}{\|\dot\gamma(t_j+0)-\dot\gamma(t_j-0)\|}\quad\left(\in L_{\sigma(j)}\right),
\end{gathered}
\tag{3.2}
$$

$j=1,\dots,m$. Here $P_i$ denotes ($i=\sigma(j)$) the orthogonal projection of $\mathbb{R}^d$ onto $L_i$. In this representation of the Euclidean path $\gamma$ we are not at all bothered by the facts that

(a) the cylinders $a_j+C_{\sigma(j)}$ ($j=1,\dots,m$) may intersect each other, or

(b) the path $\gamma$ itself may pass through certain cylinders $a_j+C_{\sigma(j)}$ without collision.

because our investigation of Euclidean paths $\gamma$ (with a fixed symbolic collision sequence $\Sigma$) will be a local, geometric analysis.

It is clear from 3.1 that the whole Euclidean path $\gamma = \gamma(\Sigma)$ is fully determined by the following data:

(i) the symbolic collision sequence $\Sigma = (\sigma(1), \dots, \sigma(m)) \in \{1, \dots, k\}^m$;

(ii) the translation vectors $a_j \in L_{\sigma(j)}$, $j = 1, \dots, m$;

(iii) and by the initial (unit) velocity $V_0 = \dot\gamma(0) \in \mathbb{S}^{d-1}$.

Therefore, the set $\Gamma = \Gamma(\Sigma)$ of all Euclidean paths $\gamma = \gamma(\Sigma)$ is naturally embedded into the product manifold $\mathbb{S}^{d-1} \times \prod_{j=1}^m L_{\sigma(j)}$ as an open submanifold by the mapping

$$\Psi : \Gamma \to \mathbb{S}^{d-1} \times \prod_{j=1}^m L_{\sigma(j)},$$

$$\Psi(\gamma) = (\dot\gamma(0); a_1(\gamma), \dots, a_m(\gamma)).$$

In this way $\Gamma$ inherits a real analytic manifold structure from the ambient space $\mathbb{S}^{d-1} \times \prod_{j=1}^m L_{\sigma(j)}$. Set

$$\begin{gathered}\Gamma(\Sigma, \vec a) = \Gamma(\Sigma, a_1, \dots, a_m) = \\ \left\{\gamma \in \Gamma : \exists\, a \in \mathbb{R}^d \text{ such that } a_j(\gamma) - a_j = P_{\sigma(j)}(a) \text{ for } j = 1, \dots, m\right\},\end{gathered} \tag{3.3}$$

($a_j \in L_{\sigma(j)}$ are given) as the closed submanifold of $\Gamma$ corresponding to the given relative positions of the cylinders $a_j + C_{\sigma(j)}$. It is easy to see that (if $\Gamma(\Sigma, \vec a) \neq \emptyset$) $\Gamma(\Sigma, \vec a)$ is a closed submanifold of $\Gamma(\Sigma)$ whose dimension is $2d-1-\dim\left(\cap_{j=1}^m A_{\sigma(j)}\right)$. *Throughout the paper we will only consider non-empty submanifolds* $\Gamma(\Sigma, \vec a)$.

We need to introduce a special family of small perturbations of Euclidean paths $\gamma \in \Gamma(\Sigma)$ corresponding to the pure spatial translations of the initial phase point, see also Section 3 of [S-Sz(1998)], especially formula (3.16) and its vicinity. Since we have now the convention $\gamma(0) = 0$, instead of translating the initial position $\gamma(0)$, we translate the cylinders $a_j + C_{\sigma(j)}$ by the same vector $a \in \mathbb{R}^d$.

**Definition 3.3-a.** For $a \in \mathbb{R}^d$ ($||a||$ is small) and $\gamma \in \Gamma = \Gamma(\Sigma)$ denote by $T_a(\gamma) = \delta$ the uniquely defined element $\delta$ of $\Gamma$ for which $\dot\delta(0) = \dot\gamma(0)$ and $a_j(\delta) - a_j(\gamma) = P_{\sigma(j)}(a)$, $j = 1, \dots, m$.

In other terms, this means that we uniformly translate every scattering cylinder $a_j + C_{\sigma(j)}$ of $\gamma$ by the same vector $a \in \mathbb{R}^d$, which essentially amounts to the same thing as if we translated the initial position by the vector $-a$.

In accordance with the part "Characterization of the Positive Subspace of the Second Fundamental Form" in Section 3 of [S-Sz(1998)], we introduce the following notions:

(a) the velocity process (history) $(V_0, V_1, \dots, V_m)$ of $\gamma \in \Gamma(\Sigma)$, where $V_0 = \dot\gamma(0)$ and $V_j = \dot\gamma(t_j + 0)$, $j = 1, \dots, m$;

(b) the orthogonal reflection $h_j$ of $\mathbb{R}^d$ across the hyperplane

$$H_j = \left(\dot\gamma(t_j + 0) - \dot\gamma(t_j - 0)\right)^{\perp},$$

$j = 1, \dots, m$

Note that the translated hyperplane $\gamma(t_j) + H_j$ is just the tangent hyperplane of the boundary of the cylinder $a_j(\gamma) + C_{\sigma(j)}$ at the point of reflection $\gamma(t_j)$. The collection of all possible hyperplanes $H_j = H_j(\gamma)$ $(\supset A_{\sigma(j)})$ arising this way makes up the space $\mathcal{P}_j$, being naturally diffeomorphic to the $(\nu_j - 1)$-dimensional real projective space $\mathbb{P}^{\nu_j - 1}(\mathbb{R})$, where $\nu_j = \dim L_{\sigma(j)}$, see also Section 3 of [S-Sz(1998)]. Given an arbitrary sequence $(V_0; h_1, \dots, h_m) \in \mathbb{S}^{d-1} \times \prod_{j=1}^m \mathcal{P}_j$, one naturally defines the velocities $V_j = V_0 \cdot h_1 \cdot \ldots \cdot h_j$ $(j = 0, \dots, m)$, i. e. the image of $V_0$ under the composite action $h_1 \cdot \ldots \cdot h_j$ of the reflections $h_1, \dots, h_j$. (Here, by convention, the reflection $h_1$ is to be applied first.) Set

$$\Phi\left(V_0; h_1, \dots, h_m\right) = V_m = V_0 \cdot h_1 \cdot \ldots \cdot h_m, \tag{3.3 – b}$$

cf. (3.17) of [S-Sz(1998)].

Let us observe that in the current representation of the Euclidean path $\gamma \in \Gamma(\Sigma)$ with $\gamma(0) = 0$, the notion of the neutral space $\mathcal{N}_0(\gamma) = \mathcal{N}(\gamma)$ (cf. definition 2.1 in [S-Sz(1998)]) is redefined as follows:

$$\mathcal{N}(\gamma) = \left\{a \in \mathbb{R}^d : \exists\, \delta > 0 \text{ such that } \forall\, \epsilon \in (-\delta, \delta)\;\; V_m\left(T_{\epsilon a}(\gamma)\right) = V_m(\gamma)\right\}. \tag{3.4}$$

For any vector $a \in \mathbb{R}^d$ and any Euclidean path $\gamma \in \Gamma(\Sigma)$ we introduce the following derivative:

$$\begin{aligned} \partial_a V_m = (\partial_a V_m)(\gamma) = \\ \lim_{\epsilon \to 0} \epsilon^{-1} \cdot \left(V_m\left(T_{\epsilon a}(\gamma)\right) - V_m\left(\gamma\right)\right). \end{aligned} \tag{3.5}$$

In accordance with the notations of Proposition 3.18 of [S-Sz(1998)], the subspace

$$\mathcal{W}_+ = \mathcal{W}_+(\gamma) = \left\{(\partial_a V_m)(\gamma) :\; a \in \mathbb{R}^d\right\} \tag{3.6}$$

is precisely the positive subspace of the second fundamental form $W$ of the image $S^t(B)$ $(t > t_m)$ of the parallel "beam of light"

$$B = \left\{x = (q, v_0) \in \mathbb{R}^d \times \mathbb{R}^d :\; v_0 = v_0(\gamma),\; q \perp v_0,\; ||q|| < \epsilon\right\}$$

under the action $S^t(\,.\,)$ of the Euclidean cylindric billiard flow determined by the cylinders $a_j(\gamma) + C_{\sigma(j)}$ generating the collisions near time moments $t_j = t_j(\gamma)$, see also formula (3.16) and the accompanying text in [S-Sz(1998)]. It is well known that the second fundamental form $W$ is symmetric and positive semi-definite, thus we get

**Proposition 3.7.** The orthogonal complement $(\mathcal{W}_+(\gamma))^\perp$ of $\mathcal{W}_+(\gamma)$ is equal to the image $\mathcal{N}_0(\gamma) \cdot h_1 \cdot \ldots \cdot h_m$ of the neutral space under the composite action $h_1 \cdot \ldots \cdot h_m$ of the reflections $h_j = h_j(\gamma)$.

Besides the positive subspace $\mathcal{W}_+(\gamma)$ we will need to use another subspace of $\mathbb{R}^d$ associated with $\gamma$. Let us consider an arbitrary vector $\vec{b} = (b_1, \dots, b_m) \in \prod_{j=1}^m L_{\sigma(j)}$. For any Euclidean path $\gamma \in \Gamma(\Sigma)$ we denote by $\delta = \mathcal{T}_{\vec b}(\gamma) \in \Gamma(\Sigma)$ the

uniquely determined Euclidean path for which $V_0(\delta) = V_0(\gamma)$ and $a_j(\delta) - a_j(\gamma) = b_j$, $j = 1, \dots, m$. In other words, the perturbed path $\delta$ corresponds to the translations of the cylinders $a_j(\gamma) + C_{\sigma(j)}$ by the vectors $b_j \in L_{\sigma(j)}$. We note here that — since our analysis of Euclidean paths is local — we are only interested in *small* perturbations $\mathcal{T}_{\vec{b}}$, so that no problem arises concerning of the smoothness of the Euclidean cylindric billiard flow.

Set

$$\partial_{\vec{b}} V_m(\gamma) = \lim_{\epsilon \to 0} \epsilon^{-1} \cdot \left[ V_m \left( \mathcal{T}_{\epsilon \vec{b}}(\gamma) \right) - V_m(\gamma) \right],$$

$$\tilde{\mathcal{W}}_+(\gamma) = \left\{ \partial_{\vec{b}} V_m(\gamma) : \vec{b} \in \prod_{j=1}^{m} L_{\sigma(j)} \right\}. \tag{3.8}$$

It is clear that $\mathcal{W}_+(\gamma) \subset \tilde{\mathcal{W}}_+(\gamma)$ and

$$\tilde{\mathcal{W}}_+(\gamma) = \mathrm{Im} \left[ \frac{\partial \Phi}{\partial \tilde{\mathcal{P}}} \left( V_0(\gamma); h_1(\gamma), \dots, h_m(\gamma) \right) \right], \tag{3.9}$$

where the right-hand-side of (3.9) denotes the image space of the partial derivative of $\Phi : \mathbb{S}^{d-1} \times \tilde{\mathcal{P}} \to \mathbb{S}^{d-1}$ with respect to the second factor $\tilde{\mathcal{P}} = \prod_{j=1}^{m} \mathcal{P}_j$, where the mapping $\Phi(V_0; h_1, \dots, h_m) = V_0 \cdot h_1 \cdot \ldots \cdot h_m$ is defined above, see also (3.17) and the paragraph preceding Proposition 3.18 in [S-Sz(1998)]. The reason why the two sides of (3.9) coincide is that, by independently translating the cylinders $a_j(\gamma) + C_{\sigma(j)}$ ($j = 1, \dots, m$) one-by-one by the vectors $\epsilon \cdot b_j$, we can independently and arbitrarily perturb the reflections $h_j = h_j(\gamma)$, as well. This argument immediately proves

**Proposition 3.10.** The mapping

$$\Theta : \Gamma(\Sigma) \to \mathbb{S}^{d-1} \times \prod_{j=1}^{m} \mathcal{P}_j,$$

defined by $\Theta(\gamma) = (V_0(\gamma); h_1(\gamma), \dots, h_m(\gamma))$ is a submersion (i. e. its derivative is surjective at every point) and, hence, it is an open mapping. □

We cite here the fundamental assertion of Proposition 3.18 from [S-Sz(1998)]:

**Proposition 3.11.** For every Euclidean path $\gamma \in \Gamma(\Sigma)$ the subspaces $\mathcal{W}_+(\gamma)$ and $\tilde{\mathcal{W}}_+(\gamma)$ are equal. □

**Remark.** Observe that — although Proposition 3.18 of [S-Sz(1998)] was originally formulated and proven for paths of cylindric billiards in a torus, the entire proof obviously carries over to the Euclidean case without any significant change.

Let us introduce now the following, useful notions of typical dimensions:

$$\Delta(\Sigma) = \max_{\gamma \in \Gamma(\Sigma)} \dim \mathcal{W}_+(\gamma) = \max_{\gamma \in \Gamma(\Sigma)} \dim \tilde{\mathcal{W}}_+(\gamma) =$$

$$\max \left\{ \dim \mathrm{Im} \left[ \frac{\partial \Phi}{\partial \tilde{\mathcal{P}}} \left( V_0; h_1, \dots, h_m \right) \right] : (V_0; h_1, \dots, h_m) \in \mathbb{S}^{d-1} \times \prod_{j=1}^{m} \mathcal{P}_j \right\}, \tag{3.12}$$

$$\Delta(\Sigma, \vec{a}) = \Delta(\Sigma; a_1, \dots, a_m) = \max \{\dim \mathcal{W}_+(\gamma) : \gamma \in \Gamma(\Sigma; a_1, \dots, a_m)\} . \tag{3.13}$$

For the definition of the non-empty, closed submanifold $\Gamma(\Sigma; a_1, \dots, a_m)$, see also (3.3) above. We note that in the first equation of (3.12) we used Proposition 3.11, while in the second equation of (3.12) we took advantage of (3.9) and Proposition 3.10.

The simple proof of the next proposition uses a quite common algebraic argument.

**Proposition 3.14.** There exist three open sets with full measure $\mathcal{O}_1 \subset \Gamma(\Sigma)$, $\mathcal{O}_2 \subset \mathbb{S}^{d-1} \times \tilde{\mathcal{P}}$, and $\mathcal{O}_3 \subset \Gamma(\Sigma; a_1, \dots, a_m)$ such that

(i) $\dim \mathcal{W}_+(\gamma) = \Delta(\Sigma)$ for every $\gamma \in \mathcal{O}_1$,

(ii)

$$\dim \mathrm{Im} \left[ \frac{\partial \Phi}{\partial \tilde{\mathcal{P}}} (V_0; h_1, \dots, h_m) \right] = \Delta(\Sigma)$$

for every $(V_0; h_1, \dots, h_m) \in \mathcal{O}_2$, and

(iii) $\dim \mathcal{W}_+(\gamma) = \Delta(\Sigma; a_1, \dots, a_m)$ for every $\gamma \in \mathcal{O}_3$.

**Proof.** We will only present here a brief sketch of the proof for the first statement, for the arguments proving the other two are analoguous.

The openness of $\mathcal{O}_1 \subset \Gamma(\Sigma)$ follows from the continuous dependence of the linear generators $\partial_{e_i} V_m(\gamma)$ ($i = 1, \dots, d$; $e_i$ is the $i$-th standard unit vector in $\mathbb{R}^d$) on $\gamma$, in other words, it follows from the lower semi-continuity of the dimension function $\dim \mathcal{W}_+(\gamma)$.

The fact that the open set $\mathcal{O}_1 \subset \Gamma(\Sigma)$ has full measure in $\Gamma(\Sigma)$ (more precisely: its complement is a countable union of smooth, proper submanifolds of $\Gamma(\Sigma)$) follows from the following observations: The coordinates of the linear generators $\partial_{e_i} V_m(\gamma)$ ($i = 1, \dots, d$) of the space $\mathcal{W}_+(\gamma)$ are algebraic functions of the coordinates of

$$\gamma = (V_0(\gamma); a_1(\gamma), \dots, a_m(\gamma)) \in \mathbb{S}^{d-1} \times \prod_{j=1}^{m} L_{\sigma(j)}.$$

These algebraic functions only contain constants, rational operations (field operations), and square roots. The reason why this is indeed so comes from the similar algebraic nature of the cylindric billiard dynamics: We are dealing with circular cylinders as scatterers. Therefore, the kinetic data of the process $\gamma$ itself (i. e. the time moments $t_j = t_j(\gamma)$, the positions $\gamma(t_j)$, and the velocities $V_j = \dot{\gamma}(t_j + 0)$) are also algebraic functions of the above type of initial variables $V_0(\gamma)$ and $a_j(\gamma)$. Recall that the time moment $t_j$ is iteratively determined by the earlier kinetic variables as the smaller root $\tau$ of the quadratic equation

$$\left\| P_{\sigma(j)} \left[ \gamma(t_{j-1}) + (\tau - t_{j-1}) \dot{\gamma}(t_{j-1} + 0) - a_j \right] \right\|^2 = r^2_{\sigma(j)}, \tag{3.15}$$

$j = 1, \dots, m$. (Here we use the natural convention $t_0 = 0$.) Note that the solutions of the equation (3.15) in the iterative process of computing the variables $t_j$, $\gamma(t_j)$

and $\dot\gamma(t_j+0)$ $(j=1,\dots,m)$ is the only point where the square root enters the whole process: all the other variables can be then expressed by rational operations. For more details, see Section 3 of [S-Sz(1999)].

Consider now the $d\times d$ matrix

$$M(\gamma)=(\partial_{e_1}V_m(\gamma),\dots,\partial_{e_d}V_m(\gamma))$$

the entries of which are algebraic functions of the coordinates of the variable

$$\gamma=(V_0(\gamma);a_1(\gamma),\dots,a_m(\gamma))\in\mathbb{S}^{d-1}\times\prod_{j=1}^m L_{\sigma(j)}.$$

The relation $\gamma\notin\mathcal{O}_1$ precisely means that $\operatorname{rank}(M(\gamma))<\Delta(\Sigma)$, i. e. every $\Delta(\Sigma)\times\Delta(\Sigma)$ sized minor of $M(\gamma)$ is zero. Since these minors are also algebraic functions of $\gamma$, and at least one of them is not identically zero (because the value $\Delta(\Sigma)$ is attained as $\operatorname{rank}(M(\gamma))$ for some $\gamma\in\Gamma(\Sigma)$), we get that the complement of $\mathcal{O}_1$ in $\Gamma(\Sigma)$ is indeed a countable union of proper, smooth submanifolds of $\Gamma(\Sigma)$. (It as an algebraic set.) $\square$

The next lemma effectively utilizes Proposition 3.14 and the theorem on mappings with constant rank from the calculus of several variables.

**Lemma 3.16.** Let $\gamma\in\mathcal{O}_1$ $(\subset\Gamma(\Sigma))$, and a small number $\epsilon_0>0$ be given. (We only study small perturbations.) Consider the following two sets of final velocities $V_m$:

$$\mathcal{V}_1=\mathcal{V}_1(\gamma,\Sigma,\epsilon_0)=\left\{V_m\left(T_a(\gamma)\right):\,a\in\mathbb{R}^d,\ ||a||<\epsilon_0\right\},$$

$$\mathcal{V}_2=\mathcal{V}_2(\gamma,\Sigma,\epsilon_0)=\left\{V_m\left(T_{\vec b}(\gamma)\right):\,\vec b=(b_1,\dots,b_m)\in\prod_{j=1}^m L_{\sigma(j)},\ \max_j||b_j||<\epsilon_0\right\}.$$

We claim that both $\mathcal{V}_1$ and $\mathcal{V}_2$ are $\Delta(\Sigma)$-dimensional, smooth manifolds containing $V_m(\gamma)$ (as an interior point), and these manifolds coincide in a neighbourhood of the point $V_m(\gamma)$.

**Proof.** Both mappings

$$a\longmapsto V_m\left(T_a(\gamma)\right)\quad(||a||<\epsilon_0)$$

and

$$\vec b\longmapsto V_m\left(T_{\vec b}(\gamma)\right)\quad(\max_j||b_j||<\epsilon_0)$$

have derivatives with constant rank $\Delta(\Sigma)$. Therefore, by the mentioned theorem on mappings with constant rank (see, for instance, Theorem 15.5, Chapter I of [H(1978)]) the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are indeed $\Delta(\Sigma)$-dimensional, smooth, embedded submanifolds of $\mathbb{R}^d$ for small enough $\epsilon_0>0$. Since $\mathcal{V}_1$ is obviously a subset of $\mathcal{V}_2$ in a neighbourhood of $V_m(\gamma)$ and these two smooth manifolds have the same dimension, they must coincide in a neighbourhood of the point $V_m(\gamma)$. $\square$

The main result of this section is

**Key Lemma 3.17.** Assume that

$$\vec{a} = (a_1, \dots, a_m) \in \prod_{j=1}^{m} L_{\sigma(j)}$$

is such a multi-vector that $\Gamma(\Sigma, \vec{a}) \neq \emptyset$. Then the typical dimensions of $\mathcal{W}_+$ in $\Gamma(\Sigma)$ and $\Gamma(\Sigma, \vec{a})$ are equal, i. e. $\Delta(\Sigma) = \Delta(\Sigma, \vec{a})$.

**Proof.** Induction on the length $m$ of $\Sigma = (\sigma(1), \dots, \sigma(m))$. For $m = 1$ the assertion is obviously true, for $\Gamma(\Sigma) = \Gamma(\Sigma, a_1)$.

Assume now that $m > 1$ and the key lemma has been proven for $m' = 1, \dots, m-1$. Consider and fix a symbolic sequence $\Sigma = (\sigma(1), \dots, \sigma(m))$ of length $m$ and a multi-vector $\vec{a} = (a_1, \dots, a_m)$ for which $\Gamma(\Sigma, \vec{a}) \neq \emptyset$.

Denote by $\Sigma'$ the truncated sequence $(\sigma(1), \dots, \sigma(m-1))$. Throughout the proof of the key lemma, for $\gamma \in \Gamma(\Sigma)$ we denote by $\gamma'$ the following, truncated Euclidean path: $\gamma'(t) = \gamma(t)$ for $0 \leq t \leq t_{m-1}(\gamma)$, and $\gamma'(t) = \gamma(t_{m-1}) + (t - t_{m-1})\dot{\gamma}(t_{m-1} + 0)$ for $t \geq t_{m-1}(\gamma)$.

Select and fix an element $\gamma_0 \in \Gamma(\Sigma, \vec{a})$. By using the induction hypothesis and the Fubini theorem, we can assume that the truncated Euclidean path $\gamma_0' \in \Gamma(\Sigma'; a_1, \dots, a_{m-1})$ belongs to the typical set $\mathcal{O}_1(\Sigma')$ of $\Gamma(\Sigma')$ and, moreover, the following additional property also holds true:

(3.18)
$$\begin{cases} \text{For almost every selection of vectors } c_j \in L_{\sigma(j)} \quad (j = 1, \dots, m) \\ \text{the Euclidean path } \delta = (V_0(\gamma); c_1, \dots, c_m) \in \Gamma(\Sigma) \text{ (if exists!) belongs to the} \\ \text{typical set } \mathcal{O}_1(\Sigma), \text{ and the truncated path } \delta' \text{ is an element of } \mathcal{O}_1(\Sigma'), \end{cases}$$

see Proposition 3.14 for the notion of the typical set $\mathcal{O}_1$.

Select and fix a small number $\epsilon_1 > 0$. Its sufficient smallness will be clarified later in the proof. There is now a perturbation $\gamma_1 \in \Gamma(\Sigma)$ of $\gamma_0$ with $V_0(\gamma_1) = V_0(\gamma_0)$ and $\|a_j(\gamma_1) - a_j(\gamma_0)\| < \epsilon_1$ $(j = 1, \dots, m)$ such that $\gamma_1 \in \mathcal{O}_1(\Sigma)$ and $\gamma_1' \in \mathcal{O}_1(\Sigma')$. We note here that the relation $V_0(\gamma_1) = V_0(\gamma_0)$ can be achieved just because of (3.18).

Consider and compare the two nearby Euclidean paths $\gamma_0'$, $\gamma_1' \in \mathcal{O}_1(\Sigma')$. Here $\gamma_0' \in \Gamma(\Sigma'; a_1, \dots, a_{m-1})$ also holds and, if the number $\epsilon_1 > 0$ has been chosen small enough, the velocity $V_{m-1}(\gamma_1') = V_{m-1}(\gamma_1)$ belongs to the small open neighbourhood $U_0 \subset \mathbb{R}^d$ of the velocity $V_{m-1}(\gamma_0') = V_{m-1}(\gamma_0)$ in which the sets $\mathcal{V}_1 = \mathcal{V}_1(\gamma_0', \Sigma', \epsilon_0)$ and $\mathcal{V}_2 = \mathcal{V}_2(\gamma_0', \Sigma', \epsilon_0)$ are $\Delta(\Sigma')$-dimensional, smooth manifolds and they coincide: $\mathcal{V}_1 \cap U_0 = \mathcal{V}_2 \cap U_0$, see Lemma 3.16. (Here we can see that the number $\epsilon_0 > 0$ should be chosen first for $\gamma_0'$, according to Lemma 3.16, and then $\epsilon_1 > 0$ must be selected small enough in order to ensure the above properties.) Now we have

$$V_{m-1}(\gamma_1) = V_{m-1}(\gamma_1') \in \mathcal{V}_1 \cap U_0 = \mathcal{V}_2 \cap U_0 \tag{3.19}$$

and, therefore, there exists a small perturbation

(3.20) $\gamma = T(\gamma) \in \Gamma(\Sigma) \quad (\|a\| < \epsilon)$

for which $V_{m-1}(\gamma_2) = V_{m-1}(\gamma_1)$. If the first selected number $\epsilon_0 > 0$ was chosen small enough then, necessarily, we have that $\gamma_2' \in \mathcal{O}_1(\Sigma')$, i. e. it is a typical Euclidean path for $\Sigma'$. Consider now the three $\Sigma'$-typical Euclidean paths $\gamma_0'$, $\gamma_1'$, $\gamma_2' \in \mathcal{O}_1(\Sigma')$. Their neutral linear spaces (measured now right after the collision $\sigma(m-1)$) are

$$\mathcal{N}(\gamma_i') = \left(\mathcal{W}_+(\gamma_i')\right)^{\perp}, \quad (i = 1, 2, 3),$$

see Proposition 3.7. By the generic nature $\gamma_i' \in \mathcal{O}_1(\Sigma')$ $(i = 1, 2, 3)$ of $\gamma_i'$ we get

$$\dim \mathcal{W}_+(\gamma_i') = \Delta(\Sigma'), \quad i = 1, 2, 3. \tag{3.21}$$

On the other hand, the space $\mathcal{W}_+(\gamma_i')$ $(i = 1, 2, 3)$ is clearly equal to the tangent space of the manifold $\mathcal{V}_1 \cap U_0 = \mathcal{V}_2 \cap U_0$ at the point $V_{m-1}(\gamma_i')$. Since $V_{m-1}(\gamma_1') = V_{m-1}(\gamma_2')$, we have that

$$\begin{cases} \mathcal{W}_+(\gamma_1') = \mathcal{W}_+(\gamma_2') \text{ and, therefore,} \\ \mathcal{N}(\gamma_1') = \mathcal{N}(\gamma_2') = \left(\mathcal{W}_+(\gamma_1')\right)^{\perp}. \end{cases} \tag{3.22}$$

The neutral space $\mathcal{N}(\gamma_i)$ of the Euclidean path $\gamma_i$ $(i = 1, 2$; the spaces $\mathcal{N}(\gamma_i)$ are now measured between the collisions $\sigma(m-1)$ and $\sigma(m))$ can be obtained obviously as the intersection

$$\mathcal{N}(\gamma_i) = \mathcal{N}(\gamma_i') \cap \left(\mathbb{R} \cdot V_{m-1}(\gamma_i) + A_{\sigma(m)}\right), \tag{3.23}$$

$i = 1, 2$. Since the right-hand-sides of (3.23) are identical for $i = 1$ and $i = 2$, we obtain that $\mathcal{N}(\gamma_1) = \mathcal{N}(\gamma_2)$ and, since $\gamma_1 \in \mathcal{O}_1(\Sigma)$ is typical with respect to $\Sigma$, we have that $\dim \mathcal{N}(\gamma_2) = \Delta(\Sigma)$, i. e. $\gamma_2 \in \mathcal{O}_1(\Sigma)$. Taking into account (3.20), we see that $\gamma_2 \in \Gamma(\Sigma; \vec{a})$, thus $\Delta(\Sigma; \vec{a}) = \Delta(\Sigma)$, as claimed. The proof of Lemma 3.17 is now complete. $\square$

**Corollary 3.24.** Suppose that $S^{[a,b]}x_0$ is a non-singular, finite trajectory segment of the genuine, toroidal, cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t \in \mathbb{R}}, \mu)$ with the collision sequence $\Sigma = (\sigma(1), \dots, \sigma(m))$ for which

(3.25)

$$\max \left\{ \dim \mathrm{Im} \left[ \frac{\partial \Phi}{\partial \tilde{\mathcal{P}}} (V_0; h_1, \dots, h_m) \right] : (V_0; h_1, \dots, h_m) \in \mathbb{S}^{d-1} \times \prod_{j=1}^{m} \mathcal{P}_j \right\} = d - 1,$$

see also (3.12). (The numbers $a$ and $b$ are supposed to be non-collision moments of time.) Then there is an open neighbourhood $U$ of $x_0$ in $\mathbf{M}$ and there is a closed, proper (i. e. of codimension at least one) algebraic set $F \subset U$ such that $S^{[a,b]}y$ is hyperbolic (sufficient) for every $y \in U \setminus F$. $\square$

**Definition 3.26.** Based upon the above corollary, we will say that the symbolic sequence $\Sigma = (\sigma(1), \dots, \sigma(m))$ is *combinatorially rich for one codimension* if (3.25) holds true.

**Corollary 3.27.** Theorem 5.1 of [S-Sz(1999)] along with Corollary 3.24 imply that *every* hard ball system $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ is completely hyperbolic! Therefore, by Pesin's theory generalized to completely hyperbolic dynamical systems with singularities [K-S(1986)], all ergodic components of a hard ball system have positive measure, and the restriction of the billiard flow to any ergodic component has the Bernoulli property, see [C-H(1996)] and [O-W(1998)]. Thus, we see that the results of the present article are stronger than the main theorem of [S-Sz(1999)] (where the complete hyperbolicity of *almost every* hard ball system was proven), despite the fact that the present approach does not use the rather involved algebraic machinery of [S-Sz(1999)].

## 4. Hyperbolicity Is Generic (Proof of the theorem)

The goal of this section is to prove that in every *transverse* cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ $\mu$-almost every phase point is hyperbolic (in other words sufficient, see Section 2). This goal will be achieved through the use of Corollary 3.24 by showing that the trajectory $S^{\mathbb{R}}x$ of almost every phase point $x \in \mathbf{M}^0 \cap \mathbf{M}^{\#}$ (for the definition of the sets $\mathbf{M}^0$ and $\mathbf{M}^{\#}$ see Section 2 of this article) contains infinitely many consecutive segments that are combinatorially rich in the sense of Definition 3.26. It turns out, however, that in proving this result the combinatorial richness described in 3.26 is not very convenient for us, so we introduce the concept of a *transitive* (or, non-splitting) symbolic sequence $\Sigma$:

**Definition 4.1.** We say that the symbolic collision sequence $\Sigma = (\sigma(1), \dots, \sigma(m))$ is *transitive* if the set of cylinders $\left\{C_{\sigma(j)} : 1 \le j \le m\right\}$ defines a transitive cylindric billiard in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ or, in other words, if the system of base spaces $\left\{L_{\sigma(j)} : 1 \le j \le m\right\}$ has the Orthogonal Non-splitting Property, see 3.1–3.6 of [S-Sz(1998)], especially 3.3–3.4 and Theorem 3.6.

The next (elementary) lemma clarifies the relationship between the transitivity of $\Sigma$ and its richness defined in 3.26.

**Lemma 4.2.** There exists an integer $C \in \mathbb{N}$ (depending merely on $d$ and the transitive collection of base subspaces $L_1, \dots, L_k \subset \mathbb{R}^d$) with the following properties: If a symbolic sequence $\Sigma = (\sigma(1), \dots, \sigma(m)) \in \{1, \dots, k\}^m$ contains at least $C$ consecutive, transitive subsequences, then the sequence $\Sigma$ is combinatorially rich as required by 3.26, i. e. formula (3.25) holds true.

**Proof.** Let $\mathcal{T}$ denote the set of all subsets $T \subset \{1, 2, \dots, k\}$ for which the collection of subspaces $\{L_i : i \in T\}$ is transitive in $\mathbb{R}^d$. Let $|\mathcal{T}| = n$ and $\mathcal{T} = \{T_1, \dots, T_n\}$. For every $T_j$ $(1 \le j \le n)$ select and fix a symbolic sequence $\Sigma^{(j)} = \left(\sigma^{(j)}(1), \dots, \sigma^{(j)}(m_j)\right)$ such that $\sigma^{(j)}(i) \in T_j$ $(i = 1, \dots, m_j)$, and $\Sigma^{(j)}$ is combinatorially rich in the sense of 3.26. Set $C = n \cdot \max_{1\le j\le n}\{m_j\}$. If a symbolic sequence $\Sigma = (\sigma(1), \dots, \sigma(m))$ fulfills the condition of the lemma with the above constant $C$, then there exists an index $j_0$ $(1 \le j_0 \le n)$, and $M = \max_{1\le j\le n}\{m_j\}$ consecutive subsegments $\Sigma_1, \dots, \Sigma_M$ of $\Sigma$ with the property that the set of labels in every $\Sigma_l$ $(1 \le l \le M)$ contains $T_{j_0}$ as a subset. It is then clear that the rich

symbolic sequence $\Sigma = (\sigma(1), \dots, \sigma(m))$ is also combinarorially rich in the sense of 3.26. $\square$

## Eventually Splitting Trajectories

**Definition 4.3.** We say that the positive semi-trajectory $S^{(0,\infty)}x$ ($x \in \mathbf{M}^0$) splits according to the non-trivial orthogonal splitting $\mathbb{R}^d = B_1 \oplus B_2$ of $\mathbb{R}^d$ if for every $t > 0$ with $S^t x \in \partial C_i$ we have $L_i \subset B_1$ or $L_i \subset B_2$.

By keeping in mind Corollary 3.24 and Lemma 4.2, in order to prove our theorem it is enough to obtain the following result, which is the analogue of Theorem 5.1 of [S-Sz(1999)].

**Main Lemma 4.4.** Assume that the cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ has a transverse system $\{L_1, \dots, L_k\}$ of base spaces. Let $\mathbb{R}^d = B_1 \oplus B_2$ be a given non-trivial orthogonal splitting of $\mathbb{R}^d$. We claim that the set

$$S_{B_1,B_2} = \left\{ x \in \mathbf{M}^0 \cap \mathbf{M}^\# : S^{(0,\infty)}x \text{ splits according to } B_1 \oplus B_2 \right\}$$

of phase points with $(B_1, B_2)$-splitting positive orbits has Liouville measure zero, i. e. $\mu\left(S_{B_1,B_2}\right) = 0$. (Note that $\mathbf{M}^0$ denotes the set of all non-singular phase points, while $\mathbf{M}^\#$ contains all phase points with infinitely many non-tangential collisions in both time directions, see also Section 2.)

**Proof.** The rest of this section will be devoted to the proof of the main lemma. The proof will be subdivided into a few lemmas.

Consider and fix an arbitrary phase point $x_0 \in S_{B_1,B_2} \setminus \partial\mathbf{M}$ ($\subset \mathbf{M}^0 \cap \mathbf{M}^\#$). We want to show that $x_0$ has an open neighbourhood $U \subset \mathbf{M} \setminus \partial\mathbf{M}$ for which $\mu\left(S_{B_1,B_2} \cap U\right) = 0$.

We set

$$I = \left\{ i : \ 1 \le i \le k, \ \exists\, t > 0 \text{ such that } S^t x_0 \in \partial C_i \right\}.$$

Plainly, $I \neq \emptyset$. By switching from $x_0$ to an image $S^t x_0$ ($t > 0$) if necessary, we can assume that for everi $i \in I$ there is an infinite sequence $t_n \nearrow \infty$ such that $S^{t_n} x_0 \in \partial C_i$ ($n \in \mathbb{N}$), i. e. the set $I$ is already stable.

Clearly, the Euclidean space $\mathbb{R}^d$ uniquely splits into an orthogonal direct sum

$$\mathbb{R}^d = \bigoplus_{j=1}^{p} E_j \oplus E_0, \tag{4.5}$$

where

(i) for $j = 1, \dots, p$ $\dim E_j \ge 2$, and the base spaces $\{L_i : i \in I, \ L_i \subset E_j\}$ enjoy the transitivity (or the Orthogonal Non-splitting Property, see the definition right before Lemma 3.3 in [S-Sz(1998)]) in $E_j$;

(ii) $\forall i \in I \ \exists j$ $(1 \le j \le p)$ such that $L_i \subset E_j$.

Since the system $\{L_i : i \in I\}$ splits, by the assumed transverseness of the entire system $\{L_1, \dots, L_k\}$ we have that there exists an index $j_0 \in \{1, \dots, k\}$ with the following property:

$$(4.6) \qquad P_{E^+}(A_{j_0}) = E^+,$$

where

$$E^+ = \bigoplus_{j=1}^{p} E_j = E_0^\perp,$$

and $P_{E^+}$ denotes the orthogonal projection of $\mathbb{R}^d$ onto $E^+$. Since $\dim A_{j_0} \le d-2$, as a consequence, we get that

$$(4.7) \qquad \dim E_0 \ge 2.$$

**Remark 4.8.** It follows easily from (i)–(ii) above that $p \ge 1$, and the linear span $\text{span}\{L_i : i \in I\}$ is equal to the space $E^+$, see also Remark 3.5 in [S-Sz(1998)]. As far as the special index $j_0$ (featuring (4.6)) is concerned, we certainly have that $j_0 \notin I$, otherwise the projection $P_{E^+}(A_{j_0})$ would be orthogonal to the space $L_{j_0} \subset E^+$.

**Definition 4.9.** The $I$-dynamics $S_I^t y$ ($y \in \mathbf{M}$, $t > 0$) is defined as follows: $S_I^t y$ evolves according to the sub-billiard system $\{C_i : i \in I\}$ in $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$, i. e. for $t > 0$ we no longer remove the cylinders $\{C_i : i \notin I\}$ from the configuration space (i. e. we no longer considering them as scatterers) but, instead, we allow for the moving point $q\left(S_I^t y\right)$ to freely pass through the transparent cylinders $C_i$ with $i \notin I$. As to the notion of sub-billiards, see subsection 2.1.

Obviously, in order to prove Main Lemma 4.4 it is enough to prove

**Proposition 4.10.** There exists an open neighbourhood $U \subset \mathbf{M} \setminus \partial\mathbf{M}$ of $x_0$ in $\mathbf{M}$ such that

$$\mu\left(\left\{y \in U \cap \mathbf{M}^0 \cap \mathbf{M}^\# : \forall\, t > 0 \quad S^t y = S_I^t y\right\}\right) = 0.$$

In the sequel we will just prove Proposition 4.10.

Fix the values of the partial kinetic energies $\varepsilon_j = \frac{1}{2}\|P_{E_j}(v)\|^2$ ($j = 1, \dots, p$) and the velocity $P_{E_0}(v)$. Introduce the notation $\left(\mathbf{M}_j, \{S_j^t\}, \mu_j\right)$ for the sub-billiard flow determined by the index set $I_j = \{i : i \in I,\ L_i \subset E_j\}$ and by the given kinetic energy $\varepsilon_j$, $j = 1, \dots, p$. The configuration space of this sub-billiard flow is naturally the torus $E_j/P_{E_j}(\mathcal{L})$ minus the intersections of the cylinders $\{C_i : i \in I_j\}$ with this torus, see also subsection 2.1. We note that the space $E_j$ here corresponds to the notation $E^+$ of 2.1.

Introduce also the notations $\mathcal{A}_j = E_j^\perp \subset \mathbb{R}^d$, $\mathcal{T}_j = \mathcal{A}_j/(\mathcal{A}_j \cap \mathcal{L})$, and $\mathcal{T}_0 = \bigcap_{j=1}^p \mathcal{T}_j$ for $j = 1, \dots, p$. Note that — as it is easy to see — the subspaces $\mathcal{A}_j$ are lattice subspaces, thus $\mathcal{T}_j$ are subtori of $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$, and $\mathcal{T}_0$ is a closed subgroup of $\mathbb{R}^d/\mathcal{L}$ being a finite extension of the subtorus $E_0/(E_0 \cap \mathcal{L})$.

**Lemma 4.11.** After fixing the values of the partial kinetic energies $\varepsilon_j = \frac{1}{2}\|P_{E_j}(v)\|^2$ ($j = 1, \dots, p$) and the velocity $P_{E_0}(v)$, there exists a natural homomorphism of dynamical systems

$$\Psi : \ (\mathbf{M}_I, \{S_I^t\}, \mu_I) \longrightarrow \prod_{j=1}^p (\mathbf{M}_j, \{S_j^t\}, \mu_j)$$

for which

(i) $\Psi$ is surjective;

(ii) two phase points $(q_1, v_1)$, $(q_2, v_2) \in \mathbf{M}_I$ are mapped to the same element by $\Psi$ if and only if $v_1 = v_2$ and $q_1 - q_2 \in \mathcal{T}_0$.

Therefore, the dynamical system $(\mathbf{M}_I, \{S_I^t\}, \mu_I)$ is locally isomorphic to the direct product

$$\prod_{j=1}^p (\mathbf{M}_j, \{S_j^t\}, \mu_j)$$

multiplied by the uniform (conditionally periodic) motion in the torus $E_0/(E_0 \cap \mathcal{L})$ with the given velocity $P_{E_0}(v)$.

**Proof.** According to subsection 2.1, the sub-billiard flow $\left(\mathbf{M}_j, \{S_j^t\}, \mu_j\right)$ is naturally a factor of $(\mathbf{M}_I, \{S_I^t\}, \mu_I)$. Denote by $\Psi_j$ the natural projection of the latter dynamical system onto the former one, $j = 1, \dots, p$. Thanks to the orthogonality of the bases of cylinders in $\mathbf{M}_{j_1}$ and $\mathbf{M}_{j_2}$ $(j_1 \neq j_2)$, we see that the $j_1$-part and $j_2$-part of the $S_I$-evolving phase point $S_I^t y_0 = y_t = (q_t, v_t)$ evolve independently. This shows that the mapping $\Psi = (\Psi_1, \dots, \Psi_p)$ with the components $\Psi_j$ is a homomorphism between the dynamical systems $(\mathbf{M}_I, \{S_I^t\}, \mu_I)$ and $\prod_{j=1}^p \left(\mathbf{M}_j, \{S_j^t\}, \mu_j\right)$. It is obvious that the mapping $\Psi$ is surjective.

The only outstanding question is (ii) in the lemma. Assume, therefore, that $\Psi(q_1, v_1) = \Psi(q_2, v_2)$. Since $P_{E_j}(v_1) = P_{E_j}(v_2)$ for $j = 0, 1, \dots, p$, we immediately have that $v_1 = v_2$. On the other hand, the equation of the $q$-components of $\Psi_j(q_1, v_1)$ and $\Psi_j(q_2, v_2)$ precisely means that $q_1 - q_2 \in \mathcal{A}_j/(\mathcal{A}_j \cap \mathcal{L}) = \mathcal{T}_j$, $j = 1, \dots, p$, i. e. $q_1 - q_2 \in \mathcal{T}_0 = \bigcap_{j=1}^p \mathcal{T}_j$. $\square$

**Proof of Proposition 4.10.** First of all, it is enough to prove 4.10 for fixed values of $\varepsilon_j = \frac{1}{2}\|P_{E_j}(v)\|^2$ $(j = 1, \dots, p)$ and the velocity $P_{E_0}(v) = v_0$. Thus, let us fix these values and prove 4.10 for the corresponding layer of the phase space.

Since for every $i \in I$ there is an infinite sequence $t_n \nearrow \infty$ such that $S^{t_n} x_0 \in \partial C_i$, by applying Lemma 4.2, property (i) after (4.5) and Corollary 3.24 for the sub-billiard factor $\left(\mathbf{M}_j, \{S_j^t\}, \mu_j\right)$ $(j = 1, \dots, p)$, we obtain that there exists an open neighbourhood $U \subset \mathbf{M} \setminus \partial\mathbf{M}$ of $x_0$ in $\mathbf{M}$ and a proper, smooth submanifold $N \subset U$ such that

$$\left\{ \begin{array}{l} \text{the sub-billiard semi-orbit } \left\{S_j^t y : \ t > 0\right\} \\ \text{is hyperbolic for every } y \in U \setminus N \text{ and } j = 1, \dots, p. \end{array} \right. \tag{4.12}$$

We note here that — although the $I$-dynamics $S_I^t y$ $(y \in U, t > 0)$ is not isomorphic to the direct product

$$\prod_{j=1}^p (\mathbf{M}_j, \{S_j^t\}, \mu_j)$$

(where $(\mathbf{M}_0, \{S_0^t\}, \mu_0)$ is the conditionally periodic motion in the torus $E_0/(E_0 \cap \mathcal{L})$), but they are still locally isomorphic according to Lemma 4.11. Therefore, in the small neighbourhood $U$ the semi-orbit $S_I^t y$ can be written as

$$S_I^t y = \left(S_0^t y_0, S_1^t y_1, \dots, S_p^t y_p\right) \tag{4.13}$$

($y \in U$, $t > 0$, $S_j^t y_j \in \mathbf{M}_j$) by using a local isomorphism provided by Lemma 4.11. Thanks to (4.12) and the generalized Pesin theory for hyperbolic dynamical systems with singularities [K-S(1986)], for $\mu$-almost every phase point $y \in U \setminus N$ and $j = 1, \dots, p$ the above component $y_j$ of $y$ belongs to an ergodic component $C^{(j)}_{\alpha_j(y)}$ of the flow $\{S_j^t\}$ with the following properties:

$$\mu_j\left(C^{(j)}_{\alpha_j(y)}\right) > 0, \tag{4.14}$$

$$S_j^t | C^{(j)}_{\alpha_j(y)} \text{ is a mixing flow.} \tag{4.15}$$

By considering generic phase points $y \in U \setminus N$, we can assume that the fixed velocity $v_0 = P_{E_0}(v)$ of the uniform motion $S_0^t y_0$ is ergodic. Let us, therefore, denote by

$$U(v_0, \varepsilon_1, \dots, \varepsilon_p, \alpha_1, \dots, \alpha_p) = U(v_0, \vec{\varepsilon}, \vec{\alpha})$$

the set of all phase points $y = (q, v) \in (U \setminus N) \cap \mathbf{M}^0$ for which $P_{E_0}(v) = v_0$, $\frac{1}{2}\|P_{E_j}(v)\|^2 = \varepsilon_j$, $\alpha_j(y) = \alpha_j$ for $j = 1, \dots, p$, and the $I$-dynamics $S_I^t y$ is non-singular (just as $S^t y$) for $t > 0$. We want to prove that

$$\mu\left(\left\{y \in U(v_0, \vec{\varepsilon}, \vec{\alpha}) : \forall\, t > 0 \quad S^t y = S_I^t y\right\}\right) = 0. \tag{4.16}$$

The direct product flow

$$\left(\mathbf{M}_0, \{S_0^t\}, \mu_0\right) \times \prod_{j=1}^{p} \left(C^{(j)}_{\alpha_j}, \{S_j^t\}, \mu_j | C^{(j)}_{\alpha_j}\right) \tag{4.17}$$

(which governs the time evolution of $S_I^t y$, $y \in U(v_0, \vec{\varepsilon}, \vec{\alpha})$, $S_0^t(y_0) = y_0 + t v_0$) is ergodic — being the product of $p$ mixing flows and an ergodic one. The condition $S^t y = S_I^t y$ ($\forall\, t > 0$) specially means that the interior of the cylinder $C_{j_0}$ (see (4.6)) is avoided. This is just the well studied phenomenon of open set (ball) avoiding! The geometric condition (4.6) means that for any given $p$-tuple of positions

$$(q_1, \dots, q_p) \in \prod_{j=1}^{p} \left(E_j / P_{E_j}(\mathcal{L})\right) \cong \prod_{j=1}^{p} \mathbb{R}^d / (\mathcal{L} + \mathcal{A}_j)$$

one can find an element $\tilde{q} \in \mathbb{R}^d / \mathcal{L}$ such that $\Psi_j(\tilde{q}) = q_j$ ($j = 1, \dots, p$) or, in other words, the natural projection

$$\pi_j:\ \mathbb{R}^d/\mathcal{L} \longrightarrow \mathbb{R}^d/(\mathcal{L}+\mathcal{A}_j) \cong E_j/P_{E_j}(\mathcal{L})$$

(see subsection 2.1) maps $\tilde{q}$ onto $q_j$, $\pi_j(\tilde{q}) = q_j$. More precisely, the geometric condition (4.6) implies that for every element $q \in \mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ there exists another element $\tilde{q} \in \mathbb{R}^d/\mathcal{L}$ for which

$$\tilde{q} \in (A_{j_0}/\mathcal{L}) + t_{j_0} \subset \mathrm{int} C_{j_0},$$

and $\tilde{q} - q \in E_0/(E_0 \cap \mathcal{L})$, i. e. even the actual connected component of the inverse image $\Psi^{-1}\left((q_1, \dots, q_p)\right)$ (to contain $\tilde{q}$) can be specified arbitrarily. (Recall that the translated subtorus $(A_{j_0}/\mathcal{L}) + t_{j_0}$ is just the axis of the cylinder $C_{j_0}$, see also the introduction.) Especially, the phase space

$$(E_0/(E_0 \cap \mathcal{L})) \times \prod_{j=1}^{p} C_{\alpha_j}^{(j)}$$

of the flow in (4.17) has an intersection of positive measure with the interior of the "forbidden" cylinder $C_{j_0}$. Therefore, due to the ergodicity of the product in (4.17), the event $\forall\, t > 0 \quad S^t y = S_I^t y$ $(y \in U(v_0, \vec{\varepsilon}, \vec{\alpha}))$ has indeed zero measure with respect to the product measure in (4.17), consequently (4.16) is true.

This finishes the proof of Proposition 4.10 and Main Lemma 4.4. $\square$

On the other hand, Main Lemma 4.4 together with Corollary 3.24 yield a proof for the theorem of this article. $\square$

**Corollary 4.18.** It follows from the generalized Pesin theory for hyperbolic dynamical systems with singularities [K-S(1986)] that every transverse cylindric billiard system has at most countably many ergodic components $C_\alpha$ (with positive measure), and the restrictions $S^t|C_\alpha$ of the flow have the Bernoulli property, see [C-H(1996)] and [O-W(1998)]

**Concluding remark.** The property of transverseness somehow means that (in rough terms) the generator spaces $A_i$ of the cylinders are big, as opposed to the condition ($A_i \cap A_j = \{0\}$ for $i \neq j$) that was assumed by P. Bálint in his Theorem 2.4 of [B(1999)]. Thus, we can say that — in some sense — the result of this article is sort of complementary to Bálint's Theorem 2.4. Out of these two result it is the present one that applies to hard ball systems.

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